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Axisymmetric internal waves generated by a travelling oscillating body

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Experiments are presented in which axisymmetric internal waves are generated by an oscillating sphere moving vertically in a stably stratified salt solution. The Reynolds numbers for the sphere based on the diameter and the mean velocity are between 10 and 200. Lighthill's theory for dispersive waves is used to calculate the phase configuration of the internal waves. The agreement between experiment and theory is reasonably good.

1. Introduction

A body moving in a stably stratified medium can produce internal gravity waves. Mowbray & Rarity (1967*b*) discuss the axisymmetric internal waves which are generated by a body moving vertically with a constant velocity in a uniformly stratified medium. They present schlieren pictures showing a 'herring bone' wave pattern behind the forcing region with no disturbances ahead of it. The wave system is stationary relative to the disturbance and the phase configuration of the waves agrees well with Lighthill's (1967) theory for dispersive waves.

In this note experiments are described in which an oscillating sphere is moved vertically in a uniformly stratified salt solution. The waves due to the oscillation are sometimes present ahead of the body and are not stationary relative to the forcing region. The axisymmetric wave patterns generated are shown to compare reasonably well with the linear theory for small amplitude waves.

2. Theoretical predictions

Lighthill's (1967) theory for waves generated in dispersive systems by travelling disturbances will be applied to this problem. We shall use cylindrical polar axes fixed in the body, with r in the horizontal plane and y positive upwards, moving with the body velocity V in the y-direction. If the wavelengths of the internal gravity waves are short compared with the density scale $|\rho_0/(d\rho_0/dy)|$, the dispersion relation takes the form (Mowbray & Rarity 1967*b*)

$$P(\omega,\alpha,\beta) = \omega^2(\alpha^2+\beta^2)-\omega_0^2\alpha^2 = 0. \tag{1}$$

ρ_0 is the density of the undisturbed fluid, α and β are the wave-numbers in the r- and y-directions and $\omega_0^2 = -(g/\rho_0)/(d\rho_0/dy)$ is the square of the Väisälä–Brunt frequency which will be considered constant. $\omega = (\omega_f + V\beta)$ is the frequency governing the direction in which energy is propagated and ω_f is the frequency of the oscillatory forcing effect. Equation (1) is written in the parametric form

$$V\beta/\omega_0 = \pm\sin\theta - N, \quad V\alpha/\omega_0 = -(V\beta/\omega_0)\tan\theta \tag{2}$$

from which the wave-number curves in figure 1 are calculated. N is the frequency ratio, ω_f/ω_0.

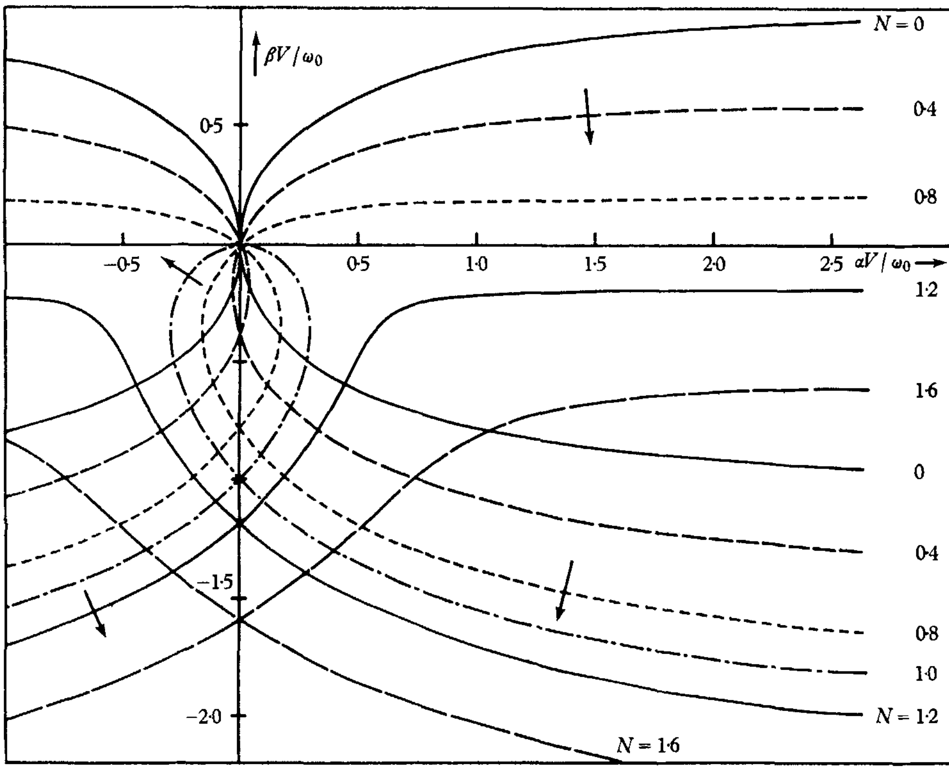

FIGURE 1. Wave-number surfaces.

Energy can only propagate away from a disturbance, which implies from Lighthill's rule that the waves which exist in a certain direction from the forcing region are those with wave-numbers corresponding to points on the wave-number surface which have normals, drawn towards higher ω, which point in that particular direction. From figure 1 it is seen that waves are not present ahead of the body when $N = 0$ which is the steady case treated by Lighthill (1967) and Mowbray & Rarity (1967*b*). When $N < 1$ the wave-number curves pass through the origin and the tangents at the origin are inclined at an angle θ to the vertical where θ satisfies $N = \pm\sin\theta$. This corresponds to the solution for an oscillating forcing region with V zero which was considered by Mowbray & Rarity (1967*a*).

The wave system is found in front of, as well as behind, the body when N is between 0 and 1·1, that is providing the body excites the relevant wave-numbers. When $N > 1{\cdot}1$ the waves due to the oscillation are restricted to a conical region behind the body and the included angle of this region will reduce as N is increased.

The locus of points of constant phase is given by $A\nabla P/(\mathbf{K}.\nabla P)$ where A is a constant, $\mathbf{K}$ is the wave-number vector and ∇ is the gradient operator in $\mathbf{K}$ space. If $B = \pm(\sin\theta - N)^{-1}$ the phase configuration evaluated from (1) takes the form

$$(r, y)\,\omega_0/(AV) = B(B\cos^3\theta, \quad B\sin\theta\cos^2\theta + 1) \tag{3}$$

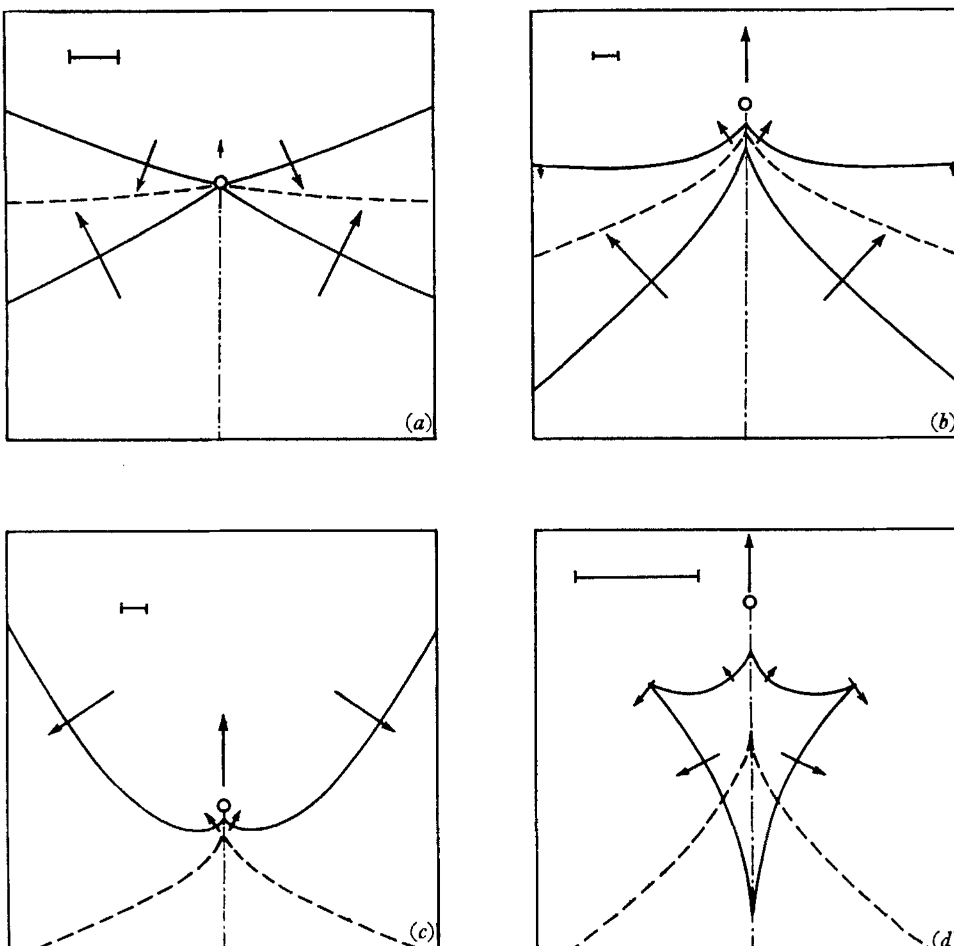

FIGURE 2. Lines of constant phase plotted as $y\omega_0/(AV)$ against $r\omega_0/(AV)$ with N equal to 0·4 in (*a*) and (*b*), 1·0 in (*c*) and 1·4 in (*d*). The arrows on the curves represent the direction and relative magnitude of the phase velocity compared with the velocity of the body, also represented by an arrow. The scale mark is of length 100 units in (*a*) and of length unity in (*b*), (*c*) and (*d*). The dashed lines correspond to the steady wave system for which $N = 0$.

which is shown in figure 2 for values of the frequency ratio N of 0·4, 1·0 and 1·4. Physically, θ is the angle which the group velocity makes with the horizontal. When $N > 1$, cusps occur at

$$(r, y)\,\omega_0/(AV) = (0, (\pm 1 - N)^{-1})$$

and at positions corresponding to

$$\sin\theta = 1{\cdot}5N - ((1{\cdot}5N)^2 - 2)^{\frac{1}{2}}.$$

When $N < 1$ cusps occur at $(r, y)\omega_0/(AV) = (0, -(1 \pm N)^{-1})$.

The waves due to the oscillation are moving relative to the undisturbed fluid with a phase velocity $V_p = \omega\mathbf{K}/(\alpha^2+\beta^2)$ or $V_p = VB\sin\theta\cos\theta$ in the direction of $\omega\mathbf{K}$. The direction and the relative magnitudes of V_p are shown by the arrows in figure 2. There is zero phase velocity where lines of constant phase have a horizontal tangent. A particular wave shape with horizontal tangents increases in size remaining geometrically similar but with the points having horizontal tangents remaining in the same horizontal plane. The waves around a body which is moving with constant velocity without oscillating do not have any horizontal tangents except at infinity.

We shall now consider the phase configuration of the waves near an oscillating body in a region with fixed physical dimensions. When the body is moving very slowly the situation corresponds to figure 2 plotted with axes having very large values of the co-ordinates. When the body moves at successively higher velocities the phase configuration of the waves we are looking at corresponds to the shapes which occur in a smaller and smaller region around the origin in figure 2. As an example, in figures 2 (*a*) and (*b*) the phase configuration for $N = 0{\cdot}4$ is shown with scale lengths of 100 units and unity respectively. Figure 2(*a*) corresponds to the small velocity and the wave shapes are obviously closer to the 'St Andrews cross waves', which were considered by Mowbray & Rarity (1967*a*), than the wave shapes in figure 2 (*b*).

Equation (3) has been used to calculate the wave patterns shown in figure 4 (*b*), (*d*), (*f*) (plate 1) and figure 5 (*b*), (*d*), (*f*) (plate 2). These figures show wave patterns for several values of the frequency ratio, N, varying between 0·70 and 1·71 for particular values of the body velocity and the Väisälä–Brunt frequency. The dashed lines in the figures are the steady system of waves corresponding to $N = 0$. Only the first few waves of each system are drawn.

3. Experiments

The glass sided tank which is 160 cm long, 90 cm high and 55 cm from front to back was filled with a stratified salt solution having a constant density gradient. Mowbray (1966) describes the way in which the tank is filled and also describes the schlieren system. The light beam passing through the tank is 46 cm in diameter. The scale of appreciable density variations, $(\rho_0/(d\rho_0/dy))$ is typically 500 cm and a typical wavelength in the experiments is of the order of 5 cm so that the waves are short. The Väisälä–Brunt frequency was assumed to be a constant in the theory whereas in the experiments it varies slightly. However, the bending of the wave crests which results from this discrepancy is small in the present experiments. The effect is discussed by Mowbray & Rarity (1967*a*).

A 2·54 cm diameter sphere is suspended with a thin nylon line from a system of pulleys and levers operated by two small electric motors so that the sphere can be drawn vertically through the stratified solution with a constant velocity on which is superimposed an oscillation of known frequency. The higher harmonics are of small amplitude. The amplitude of the oscillation is 2 cm which is rather large but this ensures that the waves due to the oscillation can be seen in the presence of the steady moving wave system.

In figure 3 the theoretical wave spacing, s, is compared with that measured from photographs and cine film. The solid line is the theoretical spacing in the direction of the cusps which lie off the axis of symmetry and the chain dotted line is the spacing along the horizontal axis, $y = 0$. The two lines meet when the cusps lie along the horizontal axis. Some points taken from the low velocity runs are

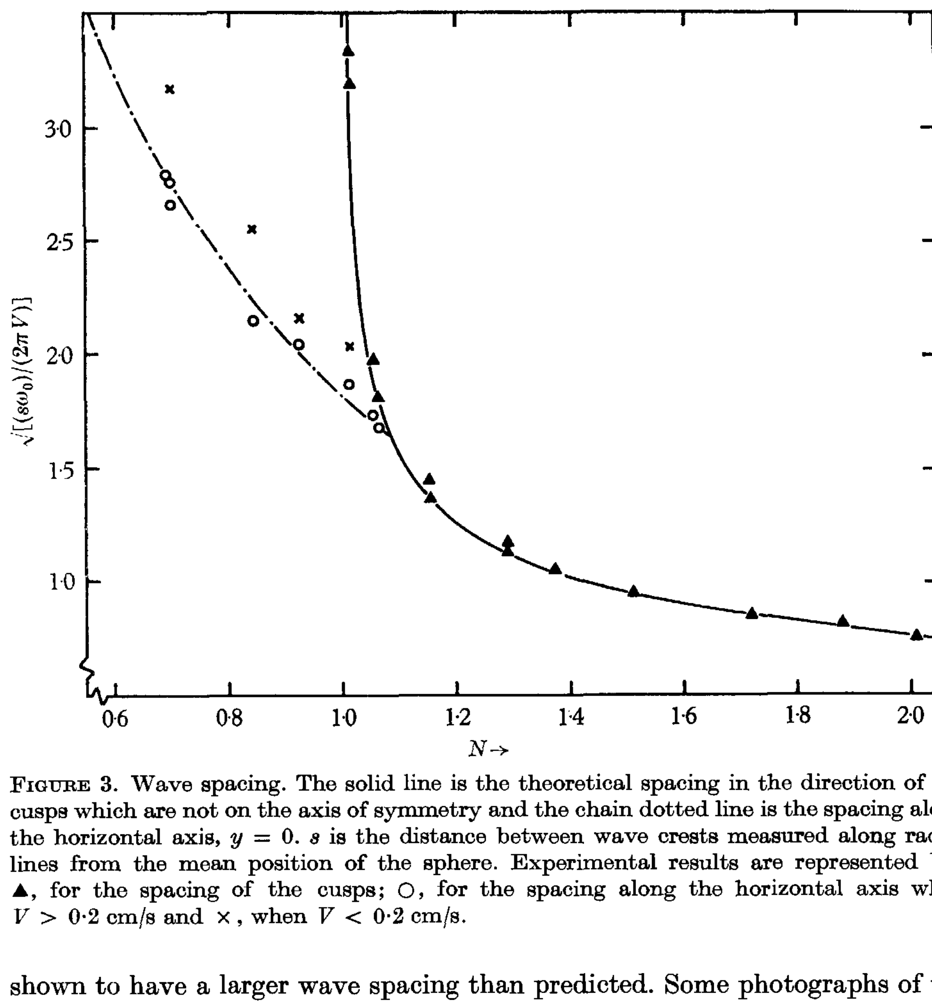

FIGURE 3. Wave spacing. The solid line is the theoretical spacing in the direction of the cusps which are not on the axis of symmetry and the chain dotted line is the spacing along the horizontal axis, $y = 0$. s is the distance between wave crests measured along radial lines from the mean position of the sphere. Experimental results are represented by: ▲, for the spacing of the cusps; ○, for the spacing along the horizontal axis when $V > 0{\cdot}2$ cm/s and ×, when $V < 0{\cdot}2$ cm/s.

shown to have a larger wave spacing than predicted. Some photographs of the wave systems are compared with the theoretical patterns in plates 1 and 2. When $N > 1$ the agreement between the theory and the experiments is very good. When $N < 1$ and the velocity of the sphere is greater than $0{\cdot}2$ cm/s the waves which sweep ahead of the sphere agree very well with theory but the waves behind the sphere tend to have a slightly larger wavelength. When $N < 1$ and $V < 0{\cdot}2$ cm/s both sets of waves have slightly larger wavelengths. Experiments using a smaller amplitude of oscillation and a smaller sphere still show the same tendency, but as the wave strength is reduced in this way it becomes difficult to see the waves with the schlieren system. However, considering the large amplitude of oscillation in the main set of experiments the agreement between the theory and the experiments is remarkably good.

In figure 4 (*a*) (plate 1) the sphere is moving with a velocity of 0·106 cm/s and this is so slow that the steady wave system is not visible. In the other photographs in plates 1 and 2 the velocity ranges between 0·29 cm/s and 0·57 cm/s and the steady wave system is clearly seen. The flow around the sphere is apparently producing all the relevant wave-numbers and frequencies because all the waves predicted by the theory are present in the experiments with one exception. In figure 4 (*e*) (plate 1) the waves which should have been almost vertical are not present because the sphere has not been travelling for a long enough distance. These portions of the waves would have been generated when the body was a very long way from the field of view. However, the body was only set in motion about 15 cm outside the field of view.

From figure 2 it is seen that the waves trailing behind the body have large phase velocities towards the body. If $V_p > V\cos\theta$ then the waves are overtaking the body and are therefore moving faster than the steady wave system. As waves can only exist in certain regions, successive wave crests eventually disappear in a manner similar to that which occurs when a body oscillates with no mean velocity.

As N is increased the kite-shaped waves decrease in size so that when N is greater than about 2·5 the waves near the body are so small that they are enclosed by the wake. There was no regular vortex shedding in the experiments and the turbulence frequencies in the wake were too high to produce a visible wave system.

4. Conclusion

The phase configuration of the axisymmetric internal waves generated around a travelling oscillating sphere has been shown to agree reasonably well with the linear theory for small amplitude waves.

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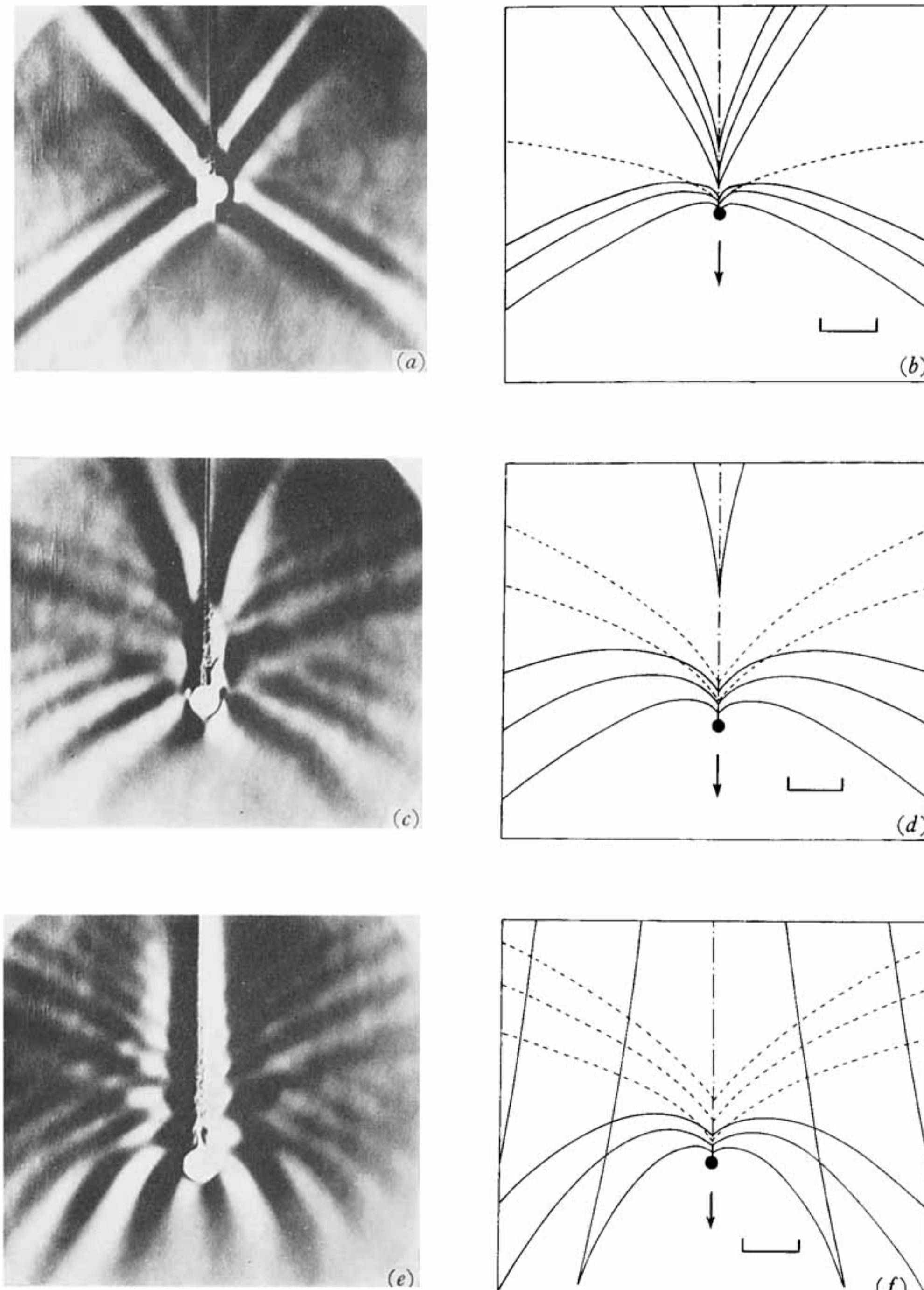

FIGURE 4. Experimental and theoretical wave patterns when $\omega_0 = 1{\cdot}14$ rad/s. The scale marks represent a length of 5 cm. (*a*) and (*b*) $V = 0{\cdot}106$ cm/s, $N = 0{\cdot}70$. (*c*) and (*d*) $V = 0{\cdot}35$ cm/s, $N = 0{\cdot}84$. (*e*) and (*f*) $V = 0{\cdot}29$ cm/s, $N = 1{\cdot}02$.

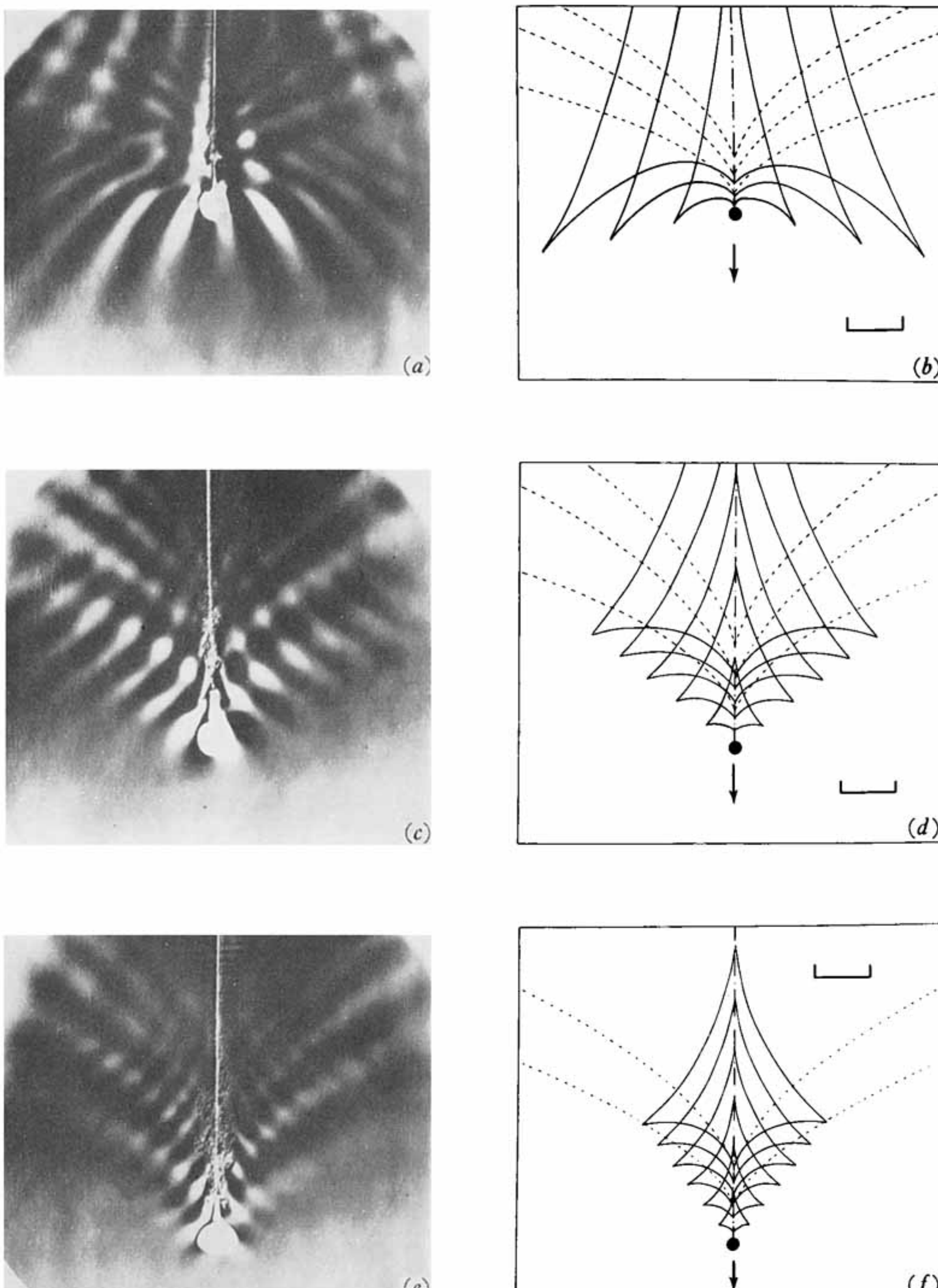

FIGURE 5. Experimental and theoretical wave patterns when $\omega_0 = 1{\cdot}14$ rad/s. The scale marks represent a length of 5 cm. (*a*) and (*b*) $V = 0{\cdot}30$ cm/s, $N = 1{\cdot}06$. (*c*) and (*d*) $V = 0{\cdot}55$ cm/s, $N = 1{\cdot}37$. (*e*) and (*f*) $V = 0{\cdot}57$ cm/s, $N = 1{\cdot}71$.